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Stability of Hierarchical Interfaces in a Random Field Model

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We study a hierarchical model for interfaces in a random-field ferromagnet. We prove that in dimension $D > 3$, at low temperatures and for weak disorder, such interfaces are rigid. Our proof uses renormalization group transformations for stochastic sequences.

KEY WORDS: Disordered systems; random-field Ising model; interfaces; hierarchical model; renormalization of stochastic sequences.

1. INTRODUCTION

One of the first questions the theory of disordered systems should be able to answer is whether and when "weak" disorder in a given model system is "relevant," that is, whether important properties of the "ordered" system change *qualitatively* when a weak random perturbation of a certain type is applied. A typical example of such a question is whether the phase structure at low temperatures[4] in, say, a ferromagnetic spin system is preserved when, e.g., random magnetic field is applied or when the exchange coupling is subjected to a random modulation (for instance, due to impurities). Notably, in the random-field Ising model (RFIM), in dimension three, this question was under strong dispute among physicists over several years, before it was resolved through rigorous mathematical proofs by Bricmont and Kupiainen[(3,4)] and Aizenman and Wehr.[(1)] A still more subtle question of considerable interest concerns non-translational-invariant Gibbs states, or *domain walls*, in these same models and, more, generally the rigidity of *interfaces* in disordered media, a question arising in a variety of contexts

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[4] Another question is the stability of critical exponents, which is even more difficult to answer.

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(for a recent review on interfaces in random media, and an extensive list of references, see, e.g., ref. 12).

There are two basic tools that are habitually used to investigate such questions: One is the so-called "replica-trick" that allows one to formally compute averages over the disorder in quenched systems by expressing them in terms of zero-component lattice field theories, thus allowing for the use of standard methods (perturbation expansions, renormalization group, etc.) from statistical mechanics. In the context of interface models in disordered media, this approach has led to the development of the "functional renormalization group" method of Fisher[10] that has led to a variety of precise predictions of scaling exponents in a large variety of models (see, e.g., refs. 14 and 18). However, there remain problems in the justification of these approaches and there has been criticism from various sides.[5,17] Another, much more simple-minded tool is the so called "Imry–Ma argument"[16] and scaling arguments based on it.[2,18] It is essentially an extension of the old Peierls argument to random systems and consists of balancing "typical" contributions to the energy and entropy of configurations coming from the random and nonrandom parts of the interactions. We will come back to this in our context in a moment. In spite of its simplicity, the Imry–Ma argument has proven extremely successful and has, in cases of dispute, shown more reliable than the more sophisticated analysis based on the replica trick (see, e.g., refs. 3 and 4 for a historic review in the RFIM context).

The appealing simplicity of the Imry–Ma argument has inspired mathematical physicists to construct rigorous proofs of its predictions. In the context of the RFIM, this has culminated, through the works of Chalker,[8] Fisher *et al.*,[11] and Imbrie,[15] in the rigorous renormalization group method of Bricmont and Kupiainen.[3,4] It is this method that will, in our opinion, provide the tools to analyze mathematically the effects of randomness in many other disordered systems.

In a recent paper Bovier and Picco[7] have applied this renormalization group method for a hierarchical model of interfaces in the random bond Ising model and proved that in dimension $D>3$ at low temperatures and for weak disorder, such interfaces are "rigid." In the present paper we want to extend this analysis to interfaces in the random-field model while at the same time streamlining the proofs.

Let us first discuss the heuristics of this situation through the Imry–Ma argument. Let us recall that the RFI Hamiltonian is given by

$$H_{\mathrm{RFI}} = -\sum_{\langle ij\rangle} s_i s_j + \sum_i \xi_i s_i \tag{1.1}$$

where the first sum is over all nearest neighbor pairs $\langle ij\rangle$ on a lattice $\mathbb{Z}^D$ and s_i are Ising spins taking the values ± 1. The random fields ξ_i are taken to be independent, identically distributed random variables (i.i.d.r.v.) with mean zero and variance σ.

In principle, one is interested in the question of whether at low temperatures there exist, in this model, pure equilibrium states in which a flat interface separates a half-space where the spins are predominantly $+1$ from its complement with predominantly -1-spins (such states are called Dobrushin states[9]). To construct such states, one considers finite volumes (for simplicity we may take D-dimensional hypercubes) and applies plus boundary conditions on the top half and minus boundary conditions on the bottom half of the cube. In the absence of the random field, the corresponding *ground state* then has all spins above the equatorial plane equal to plus one and all those below equal to minus one; there is a flat *interface* in the equatorial plane separating the plus and minus phase. It is known from the work of Dobrushin[9] and Gallavotti[13] that in $D \geqslant 3$ at low temperatures Gibbs states concentrated near this configuration exist (the interface is "stable") while in $D \leqslant 2$ at any nonzero temperature such an interface will have unbounded fluctuations (in the thermodynamic limit) and translational invariance of the Gibbs state is restored.

To predict the effect of the random field in this situation, one may try to estimate the change in energy produced by deforming the flat interface. For simplicity, we may consider a cylinder of height h and diameter L sitting on the flat interface. If we flip all the spins in this cylinder, we get two contributions to the energy: A surface energy from the "wall" of the cylinder, $E_{\text{surf}} \sim L^{D-2}h$, and a volume energy from the random field, $E_{\text{vol}} \sim \pm 2 \sum_{i \in \text{cyl}} \xi_i$, where the sign depends on whether the deformation builds into the minus or plus phase. Now the ξ_i are assumed to be independent random variables with mean zero and variance σ, so by the central limit theorem, for large cylinders the energy contribution of the random fields is "typically" of the order of $E_{\text{vol}} \sim \sigma(hL^{D-1})^{1/2}$, with arbitrary sign. Now, if $D > 3$, for any h and L large, the volume term will always be small compared to the surface term and thus this Imry–Ma argument predicts that the random fields are irrelevant in $D > 3$.

While at first glance this argument seems convincing, some more reflection shows that *in principle* it could be quite false. For, even if we are just interested in the form of the ground state, the real question is not what the energy of some, be it "typical" deformation is, but whether there exists *some* deformation, albeit a very exceptional one, that has a negative energy. In some sense, the question of interface stability is rather one of *large deviations* then of central limit theorems. Fortunately, in probability theory, these two things frequently go hand in hand. However, a serious problem one is facing when trying to analyze this situation is that the number of possible deformations of the interface (of, say, fixed height and volume) is far too large to permit one to estimate the probability that there exists one with negative energy by the sum of the corresponding probabilities. While such a bound is acceptable if one has only small deformations (where the scale of "small" depends on the variance of the random fields), on a large scale it gives divergent results. It is thus crucial to take into account the *dependence* of the random variables corresponding to different, but not very different, deformations. For the bulk phase of the RFIM, this had first been done, under some restrictions, in refs. 8 and 11, but the most natural approach to solving this problem is the renormalization group.[3,4] In the present paper we will perform such a renormalization group analysis in a somewhat simplified *hierarchical model* for the interface in the RFIM. The main virtue of this approximation is that the combinatoric aspects of the proofs simplify considerably while the main probabilistic features of the problem are preserved. We would like to note that compared to the random bond case treated in ref. 7, these present some serious complications in that correlations between random variables have to be kept track off in the renormalization process.

Let us now give a definition of the hierarchical model we want to study. In this model, the interface is given by a collection of towers and wells whose bases are formed by squares of side lengths L^n, where L is some positive integer to be chosen. More precisely, we consider a d-dimensional (we set always $d = D - 1$) square of side length L^N. We divide it into L^d squares of side length L^{N-1}, each of these into L^d squares of side length L^{N-2}, and so forth until we arrive at squares of side length one. All these squares form the potential bases of towers. The squares of the nth hierarchy (i.e., those of side length L^n) are labeled by the set

$$Y_n = \{ y \in \mathbb{Z}_+^d \mid y_i \in \{0,..., L^{N-n} - 1\} \} \tag{1.2}$$

We denote by $\mathscr{L}^{-1}$ the map from Y_n to Y_{n+1} such that $(\mathscr{L}^{-1} y)_i = \text{int}(y_i/L)$, i.e., the map that associates to $y \in Y_n$ the block in the next hierarchy that contains y. We also denote, following the usual habit, for $y \in Y_{n+1}$, by $\mathscr{L} y$ the collection of the sites $x \in Y_n$ such that $\mathscr{L}^{-1} x = y$.

A surface is described by specifying a set of heights of towers of all hierarchies,

$$\{ h_y^{(n)} \}_{y \in Y_n}^{n = 0,..., N}$$

Obviously the actual height of the surface above the point x, $H_x^{(N)}$, is given by the sum of all the heights of all towers whose base contains x, i.e.,

$$H_x^{(N)} = \sum_{n=0}^{N} h_{\mathscr{L}^{-n} x}^{(n)} \tag{1.3}$$

Note that such a surface is in particular of the solid-on-solid (SOS) type. The energy of a surface is given by

$$E_{N,J}(\{h\}) = \sum_{n=0}^{N} \sum_{y \in Y_n} |h_y^{(n)}| \, L^{(d-1)n} + \sum_{x \in Y_0} J_x(H_x^{(N)}) \tag{1.4}$$

with

$$J_x(h_x) = \begin{cases} \sum_{h=1}^{h_x} 2\xi_{x,h} & \text{if} \quad h_x \geqslant 1 \\ 0 & \text{if} \quad hx_x = 0 \\ -\sum_{h=h_x}^{-1} 2\xi_{x,h} & \text{if} \quad h_x \leqslant -1 \end{cases} \tag{1.5}$$

[Note that we use the notation $i = (x, h)$ for points $i \in \mathbb{Z}^D$.]

The partition function $Z_N(\beta, J)$ is defined as

$$Z_N(\beta, J) = \sum_{\{h\}} e^{-\beta E_{N,J}(\{h\})} \tag{1.6}$$

We also define the finite-volume Gibbs measures $\mu_{N,\beta,J}$ on the probability space of the $\{h\}$ by

$$\mu_{N,\beta,J}(\cdot) = \frac{1}{Z_N(\beta, J)} \sum_{\{h\}} \cdot e^{-\beta E_{N,J}(\{h\})} \tag{1.7}$$

A quantity of particular interest will be the expectation of the absolute value of the height of the interface at a given point x,

$$m_x(\beta, J) \equiv \limsup_{N \uparrow \infty} \mu_{N,\beta,J}(|H_x^{(N)}|) \tag{1.8}$$

The explicit construction of the infinite-volume Gibbs measure and an investigation of its formal structure will be presented in a follow-up paper.[6]

This model is the same as the one considered in ref. 7; the only differences are the assumptions on the random variables $J_x(H)$. In the random bond situation one was interested essentially in $J_x(H)$ that were i.i.d.r.v. In fact it had been assumed that for fixed x the family $\{J_x(H)\}_{H \in \mathbb{Z}}$ forms a stationary stochastic process, and that these processes are independent for different values of x. Moreover, ref. 7 assumed Gaussian bounds on the distributions, i.e.,

$$\mathbb{P}[|J_x(H)| > \delta] \leqslant e^{-\delta^2/2\sigma^2} \tag{1.9}$$

for $\delta \geqslant \sigma$, with σ small. In the random field situation, with $J_x(H)$ given by (1.3), these conditions are of course not satisfied. With $\xi_{x,h}$, say, i.i.d. Gaussian random variables, one gets Gaussian bounds on the $J_x(H)$ that depend on H,

$$\mathbb{P}[|J_x(H)| > \delta] \leqslant e^{-\delta^2/2\sigma^2 |H|} \tag{1.10}$$

and of course the processes $\{J_x(H)\}_{H \in \mathbb{Z}}$ are not stationary. It will turn out in our analysis that in this situation, it is not enough to consider the $J_x(H)$ alone, but we will have to control the stochastic processes formed by the difference variables

$$D_x(H, H') \equiv J_x(H) - J_x(H') \tag{1.11}$$

We will require two pieces of information: First, that for given x the family $\{D_x(H, H')\}_{H,H' \in \mathbb{Z}}$ forms a stochastic process that is stationary under the simultaneous shift $(H, H') \to (H+k, H'+k)$, for $k \in \mathbb{Z}$. Again these processes will be independent for different x. Second, we need bounds on the distributions of the $D_x(H, H')$ of the form

$$\mathbb{P}[D_x(H, H') > \delta] \leqslant \exp\left(-\frac{\delta^2}{2\sigma^2 |H-H'|}\right) \tag{1.12}$$

Note that (1.12) implies the same bound on $\mathbb{P}[D_x(H, H') < -\delta]$ since $D_x(H, H') = -D_x(H', H)$. These conditions are of course satisfied for i.i.d. random fields $\xi_{x,h}$ that satisfy Gaussian bounds of the form (1.9). The difficulty we will have to cope with is to preserve such conditions in the course of the renormalization process. In fact, we will see that it is impossible to maintain the purely Gaussian bounds on the probabilities, as the renormalization necessarily introduces terms that decay only exponentially with δ. It is thus natural to enlarge the class of admissible random variables from the start by adding terms $\exp(-\delta/\sigma^2)$ to the left-hand sides of (1.10) and (1.12). This then means that our results also apply in the case of exponentially distributed random fields.

The precise formulation of our main result is given in the following theorem.

Theorem 1. Let $\{J_x(H)\}_{H \in \mathbb{Z}}$, for $x \in \mathbb{Z}^d$, be independent stochastic processes such that the associated difference processes $\{D_x(H, H')\}_{H,H' \in \mathbb{Z}}$ are stationary under simultaneous shifts in H and H'. Assume, moreover, that

$$\mathbb{E}D(H, H') = 0 \qquad \text{for all} \quad H, H' \in \mathbb{Z} \tag{1.13}$$

and that for all $\delta > 0$ and all $H \neq H' \in \mathbb{Z}$,

$$\mathbb{P}[D_x(H, H') > \delta] \leqslant \exp\left(-\frac{\delta^2}{2\sigma^2 |H - H'|}\right) + \exp\left(-\frac{\delta}{\sigma^2}\right) \quad (1.14)$$

Then for $d > 2$, there exist $\beta_0 < \infty$, $\sigma_0 > 0$, and $L_0 < \infty$ and finite positive constants c_1 and c_2 such that for all $\beta \geqslant \beta_0$, $\sigma \leqslant \sigma_0$, and $L \geqslant L_0$,

$$\mathbb{P}[m_x(\beta, J) > \delta] \leqslant \min\left\{\exp\left(-\frac{1}{\tilde{\sigma}^2}\right), \exp\left(-\frac{\delta}{\tilde{\sigma}^2}\right)\right\} \quad (1.15)$$

for all $\delta > e^{-c_2\beta}$ and for all $x \in \mathbb{Z}^d$, where $\tilde{\sigma} = c_1 \sigma$.

Remark 1. Note that we give conditions only on the difference processes of the $J_x(H)$. But these specify the J_x up to an H-independent random variable which has no effect on the thermodynamics and therefore can conveniently be chosen as to make $J_x(0) \equiv 0$.

Remark 2. The restriction of the range of the δ's for which (1.16) holds takes into account thermal fluctuations that are present also in the absence of disorder. Notice that (1.15) implies that

$$\mathbb{E}m_x(\beta, J) \leqslant O\left(\max\left\{\exp\left(-\frac{1}{\tilde{\sigma}^2}\right), \exp(-c_2\beta)\right\}\right) \quad (1.16)$$

The remainder of this paper is organized as follows: In Section 2 we derive the renormalization group transformations and the resulting formulas for $m_x(\beta, J)$. In Section 3 we analyze these transformations in the case $\beta = \infty$, which is more transparent and serves to illustrate the main probabilistic aspects of this problem, and prove a corresponding weaker form of Theorem 1. In Section 4 the full proof of Theorem 1 is presented.

2. THE RENORMALIZATION GROUP TRANSFORMATION

In this section we derive the formal tools to analyze our model, namely the renormalization group transformations. To do so, as in the random bond model,[7] we compute the partition function (1.7) by a successive summation over the different hierarchies of towers. This maps the model with N hierarchies to an identical one with $N-1$ hierarchies with a renormalized temperature and a renormalized probability distributions for the J's. We recall the first steps from ref. 7:

$$\begin{aligned} Z_N(\beta, J) &= \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(1)}\}} \exp\left\{-\beta \sum_{n=1}^{N} \sum_{y_n} |h_{y_n}^{(n)}| L^{(d-1)n}\right\} \\ &\quad \times \sum_{\{h_x^{(0)}\}} \exp\left\{-\beta\left[\sum_x (|h_x^{(0)}| + J_x(H_x^{(N)}))\right]\right\} \\ &= \sum_{h_y^{(N-1)}} \cdots \sum_{h_y^{(0)}} \exp\left\{-\beta L^{(d-1)}\left[\sum_{n=0}^{N-1} |h_{y_n}^{(n)}| L^{(d-1)n} + \sum_y \tilde{J}_y(H_y^{(N-1)})\right]\right\} \\ &= Z_{N-1}(\beta^{(1)}, \tilde{J}) \end{aligned} \tag{2.1}$$

where

$$\tilde{J}_y(H) = \frac{-1}{\beta L^{d-1}} \sum_{x \in Ly} \ln\left[\sum_h \exp\{-\beta(|h| + J_x(H+h))\}\right] \tag{2.2}$$

and

$$\beta^{(1)} = \beta L^{d-1} \tag{2.3}$$

We find it convenient to introduce the function Φ_β: $\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ by

$$\Phi_\beta(\{a_n\}_{n\in\mathbb{Z}}) = -\frac{1}{\beta} \ln \sum_{n\in\mathbb{Z}} e^{-\beta a_n} \tag{2.4}$$

Note that Φ_β has the natural physical interpretation of the free energy of the sequence $\{a_n\}$. Using this definition, we can write (2.2) as

$$\tilde{J}_y(H) = \frac{1}{L^{d-1}} \sum_{x\in Ly} \Phi_\beta(\{|h| + J_x(H+h)\}_{h\in\mathbb{Z}}) \tag{2.5}$$

As in the case of the random bond model, it is easy to show that under the conditions of Theorem 1, the $\tilde{J}_y(H)$ are well-defined, almost surely finite random variables. [Note that this is also true for the partition function $Z_N(\beta, J)$ itself.] A crucial step in ref. 7 was to subtract from the $\tilde{J}_y(H)$ their mean, which, due to the assumption of stationarity of the original random variables, *was independent of H*. This condition is not satisfied in our case, and at first glance it does not appear obvious that the $\tilde{J}_y(H)$ have an H-independent mean (note that their distribution depends strongly on H!). If this were indeed so, we could not carry out our renormalization program, since these means would grow without bounds. However, the assumption made in Theorem 1 that the difference process $D_x(H, H')$ associated to the $J_x(H)$ be stationary with respect to diagonal shifts will guarantee that the means of the renormalized $\tilde{J}$ are H independent. This is one of the reasons we have to control not only the renormalization of the J, but also, and in particular, that of their differences.

Note that from (2.2) we obtain easily the following equation for the renormalized $D_y^{(1)}(H, H') \equiv \tilde{J}_y(H) - \tilde{J}_y(H')$:

$$D_y^{(1)}(H, H') = \frac{1}{L^{d-1}} \sum_{x \in Ly} \{D_x(H, H') + \Phi_\beta(\{|h| + D_x(H+h, H)\}_{h \in \mathbb{Z}}) - \Phi_\beta(\{|h| + D_x(H'+h, H')\}_{h \in \mathbb{Z}})\} \tag{2.6}$$

One sees that by the assumption of stationarity for the $D(H, H')$, $D_y^{(1)}(H, H')$ has expectation equal to zero. It is thus convenient to define

$$J_y^{(1)}(H) \equiv D_y^{(1)}(H, 0) \tag{2.7}$$

by which the energy of a flat surface in the renormalized model given by the $J_y^{(1)}(H)$ is set to be zero. The $J_y^{(1)}(H)$ now have the property to be centered, while the recursion (2.1) for the partition function takes the form

$$Z_N(\beta, J) = \exp\left\{\beta^{(1)} \sum_{y \in Y_1} \tilde{J}_y(0)\right\} Z_{N-1}(\beta^{(1)}, J^{(1)}) \tag{2.8}$$

The factor in front of Z_{N-1} on the right-hand side is of course just a (random) constant contributing to the free energy but not to expectation values.

Note that Eq. (2.6) makes manifest that the $D_y^{(1)}(H, H')$ form again a stationary process with respect to diagonal shifts, and thus all conditions to iterate this procedure will easily be seen to be satisfied, excepting of course the exponential bounds on the new distributions, whose proof will constitute the bulk of this paper. The set of recursive equations we obtain is now

$$Z_N(\beta, J) = \exp\left\{\sum_{k=1}^{n} \beta^{(k)} \sum_{y \in Y_k} \tilde{J}_y^{(k-1)}(0)\right\} Z_{N-n}(\beta^{(n)}, J^{(n)}) \tag{2.9}$$

$$\beta^{(n)} = L^{(d-1)n}\beta \tag{2.10}$$

$$D_y^{(n+1)}(H, H') = \frac{1}{L^{d-1}} \sum_{x \in \mathscr{L}y} \{D_x^{(n)}(H, H') + \Phi_{\beta^{(n)}}(\{|h| + D_x^{(n)}(H+h, H)\}_{h \in \mathbb{Z}}) - \Phi_{\beta^{(n)}}(\{|h| + D_x^{(n)}(H'+h, H')\}_{h \in \mathbb{Z}})\} \tag{2.11}$$

$$J_y^{(n)}(H) \equiv D_y^{(n)}(H, 0) \tag{2.12}$$

The meaning of the $\tilde{J}$ is similar as in the first step of the iteration. Now we can carry on as in the random bond case to get a recursion equation for the mean height. Note first that by the triangle inequality

$$\mu_{N,\beta,J}(|H_x^{(N)}|) \leqslant \sum_{n=0}^{N} \mu_{N,\beta,J}(|h_{\mathscr{L}^{-n}x}^{(n)}|) \tag{2.13}$$

By summing over the h's of the first $n-1$ hierarchies, we obtain

$$\begin{aligned} &\mu_{N,\beta,J}(|h_y^{(n)}|) \\ &= \frac{1}{Z_N(\beta, J)} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(0)}\}} |h_y^{(n)}| \\ &\quad \times \exp\left\{-\beta\left[\sum_{n=0}^{N} \sum_{y_k} |h_{y_k}^{(k)}| L^{(d-1)k} + \sum_{x \in Y_0} J_x(H_x^{(N)})\right]\right\} \\ &= \frac{1}{Z_{N-n}(\beta^{(n)}, J^{(n)})} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(n)}\}} |h_y^{(n)}| \\ &\quad \times \exp\left\{-\beta^{(n)}\left[\sum_{k=n}^{N} \sum_{y_k} |h_{y_k}^{(k)}| L^{(d-1)(k-n)} + \sum_{x \in Y_n} J_x^{(n)}(H_x^{(N-n)})\right]\right\} \end{aligned} \tag{2.14}$$

On the nth hierarchy, for $y' \neq y$, the sums over the $h_{y'}^{(n)}$ can be carried out in the same way as before, but the sum over $h_y^{(n)}$ has to be treated separately. Defining

$$\langle |h_y^{(n)}| \rangle_n (H) \equiv \frac{\sum_h (\exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\})\,|h|}{\sum_h \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\}} \tag{2.15}$$

we get

$$\begin{aligned} \mu_{N,\beta,J}(|h_y^{(n)}|) = &\frac{1}{Z_{N-n-1}(\beta^{(n+1)}, J^{(n+1)})} \\ &\times \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(n+1)}\}} \langle |h_y^{(n)}| \rangle_n (H_y^{(N-n-1)}) \\ &\times \exp\left\{-\beta^{(n+1)} \sum_{k=n+1}^{N} \sum_{y_k} |h_{y_k}^{(k)}| L^{(d-1)(k-n-1)}\right. \\ &\left. + \sum_{x \in Y_{n+1}} J_x^{n+1}(H_x^{(N-n-1)})\right\} \end{aligned} \tag{2.16}$$

Continuing to sum over the hierarchies as before, we obtain finally that

$$\mu_{N,\beta,J}(|h_y^{(n)}|) = \langle |h_y^{(n)}| \rangle_N (0) \tag{2.17}$$

where for $m > n$, $\langle |h_y^{(n)}| \rangle_m (H)$ is defined recursively as

$$\langle |h_y^{(n)}| \rangle_m (H) = \frac{\sum_h (\exp\{-\beta^{(m)}[|h| + J_y^{(m)}(H+h)]\}) \langle |h_y^{(n)}| \rangle_{m-1} (H+h)}{\sum_h \exp\{-\beta^{(m)}[|h| + J_y^{(m)}(H+h)]\}} \quad (2.18)$$

This recursive scheme will prove useful to convert the probabilistic bounds on the sequences of random variables $J_y^{(n)}(H)$ into bounds on the probabilities for the random variables $\mu_{N,\beta,J}(|H_x^{(N)}|)$. The main work of the next sections will thus consist in controlling the probability distributions of the stochastic processes $D_y^{(n)}(H, H')$.

3. THE RENORMALIZATION GROUP FLOW AT ZERO TEMPERATURE

As a first step we now treat the case where $\beta = \infty$. Here the function Φ_β introduced in Section 2.4 simply becomes the infimum, i.e.,

$$\lim_{\beta \to \infty} \Phi_\beta(\{a_n\}_{n \in \mathbb{Z}}) = \inf_{n \in \mathbb{Z}} (a_n) \quad (3.1)$$

provided only $\{a_n\}$ is such that $\Phi_\beta(\{a_n\}_{n \in \mathbb{Z}}) > -\infty$ for β large enough, a condition that will be satisfied almost surely for all sequences appearing in the sequal. This makes the renormalization group transformations particularly transparent: Equation (2.11) thus may be written in the form

$$D_y^{(n+1)}(H, H') = \frac{1}{L^{d-1}} \sum_{x \in \mathcal{L}y} \{D_x^{(n)}(H, H') + \inf_{h \in \mathbb{Z}} (|h| + D_x^{(n)}(H+h, H)) - \inf_{h \in \mathbb{Z}} (|h| + D_x^{(n)}(H'+h, H'))\} \quad (3.2)$$

These recursions have the following property:

Proposition 3.1. Let $\{D_x^{(n)}(H, H')\}_{H, H' \in \mathbb{Z}}$ be difference processes, i.e., let for all H, H', $H'' \in \mathbb{Z}$

$$D_x^{(n)}(H, H') + D_x^{(n)}(H', H'') = D_x^{(n)}(H, H'') \quad (3.3)$$

let the processes be independent for different x, stationary with respect to the diagonal shift $(H, H') \to (H+k, H'+k)$ for $k \in \mathbb{Z}$, $\mathbb{E} D_x^{(n)}(H, H') = 0$, and satisfy, for all $\delta > 0$,

$$\mathbb{P}[D_x^{(n)}(H, H') < -\delta] \leqslant \exp\left(-\frac{\delta^2}{2\sigma_n^2 |H - H'|}\right) + \exp\left(-\frac{\delta}{\sigma_n^2}\right) \quad (3.4)$$

Then, $\{D_x^{(n+1)}(H, H')\}_{H,H'\in\mathbb{Z}}$ is again a difference process, stationary with respect to diagonal translation, $\mathbb{E}D_x^{(n+1)}(H, H')=0$, and, for all $d>2$, if L is sufficiently large and σ_n small enough, there exists a constant $c<1$, independent of σ_n, such that with $\sigma_{n+1}=c\sigma_n$, for all $\delta>0$,

$$\mathbb{P}[D_x^{(+n+1)}(H, H')< -\delta]\leqslant \exp\left(-\frac{\delta^2}{2\sigma_{n+1}^2\,|H-H'|}\right) + \exp\left(-\frac{\delta}{\sigma_{n+1}^2}\right) \tag{3.5}$$

Proof. Stationarity of the process $D_x^{(n+1)}(H, H')$ is obvious, since the recursion (3.2) is of the form [use (3.3)]

$$\begin{aligned} D_y^{(n+1)}(H, H') &= \frac{1}{L^{d-1}} \sum_{x\in\mathscr{L}y} \{D_x^{(n)}(H, H') + \inf_{h\in\mathbb{Z}} (|h| + D_x^{(n)}(H+h, H')) \\ &\quad - \inf_{h\in\mathbb{Z}} (|h| - D_x^{(n)}(H, H'+h))\} \\ &= F(\{D_x^{(n)}(H+h, H'+h')\}_{h,h'\in\mathbb{Z},x\in\mathscr{L}y}) \end{aligned} \tag{3.6}$$

with F a measurable function from $\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{R}$ (see, e.g., ref. 19). Moreover, using the stationarity of the $D_x^{(n)}(H, H')$, one sees that

$$\begin{aligned} \mathbb{E}D_x^{(n+1)}(H, H') &= \mathbb{E}D_x^{(n)}(H, H') + \mathbb{E}\inf_{h\in\mathbb{Z}} (|h| + D_x^{(n)}(H+h, H)) \\ &\quad - \mathbb{E}\inf_{h\in\mathbb{Z}} (|h| + D_x^{(n)}(H'+h, H')) \\ &= 0 \end{aligned} \tag{3.7}$$

The fact that $D_x^{(n+1)}(H, H')$ is a difference process also is a trivial consequence of the induction hypothesis and the structure of Eq. (3.2).

The real task of the proof is thus to show the bounds (3.5) for the renormalized process. Note that by stationarity it is enough to consider $D_y^{(n+1)}(H, 0)\equiv J_y^{(n+1)}(H)$. We rewrite (3.2) for them in the form

$$\begin{aligned} J_y^{(n+1)}(H) &= \frac{1}{L^{d-1}} \sum_{x\in\mathscr{L}y} \inf_{h\in\mathbb{Z}} (|h| + J_x^{(n)}(H+h)) \\ &\quad - \frac{1}{L^{d-1}} \sum_{x\in\mathscr{L}y} \inf_{h\in\mathbb{Z}} (|h| + J_x^{(n)}(h)) \end{aligned} \tag{3.8}$$

Now

$$\inf_{h\in\mathbb{Z}} (|h| + J_x^{(n)}(h)) \leqslant J_x^{(n)}(0) \equiv 0$$

so that we may work with the bound

$$J_y^{(n+1)}(H) \geqslant \frac{1}{L^{d-1}} \sum_{x \in \mathscr{L}y} \inf_{h \in \mathbb{Z}} (|h| + J_x^{(n)}(H+h)) \tag{3.9}$$

Our proof will consist of two steps: First we show that a bound of the form (3.4), with slightly enlarged σ_n, for $\delta > \sigma_n |H|^{1/2}$, holds for the variables

$$I_x(H) = \inf_{h \in \mathbb{Z}} (|h| + J_x^{(n)}(H+h)) \tag{3.10}$$

We then show that such bounds extend to all positive values of δ and that summing over independent $I_x(H)$ and dividing by the factor L^{d-1} finally reduces the variances in the desired way, which yields (3.5).

We start by separating the contributions of "small" and "large" h's in the infimum,

$$\mathbb{P}[I_x(H) < -\delta] \leqslant \mathbb{P}[\inf_{|h| \geqslant |H|} (|h| + J_x^{(n)}(H+h)) < -\delta] + \mathbb{P}[\inf_{|h| < |H|} (|h| + J_x^{(n)}(H+h)) < -\delta] \tag{3.11}$$

The first term in (3.11) causes no problems:

$$\begin{aligned} &\mathbb{P}[\inf_{|h| \geqslant |H|} (|h| + J_x^{(n)}(H+h)) < -\delta] \\ &\leqslant \sum_{|h| \geqslant |H|} \mathbb{P}[|h| + J_x^{(n)}(H+h) < -\delta] \\ &\leqslant \sum_{|h| \geqslant |H|} \left[\exp\left(-\frac{(\delta + |h|)^2}{2\sigma_n^2 |H+h|}\right) + \exp\left(-\frac{\delta + |h|}{\sigma_n^2}\right)\right] \end{aligned} \tag{3.12}$$

Using that for $|h| \geqslant |H|$

$$\begin{aligned} \exp\left(-\frac{(\delta + |h|)^2}{2\sigma_n^2 |H+h|}\right) &\leqslant \exp\left(-\frac{2\delta |h| + |h|^2}{2\sigma_n^2(|H| + |h|)}\right) \\ &\leqslant \exp\left(-\frac{\delta}{\sigma_n^2} - \frac{|h|}{4\sigma_n^2}\right) \end{aligned} \tag{3.13}$$

we obtain for (3.11)

$$\begin{aligned} &\mathbb{P}[\inf_{|h| \geqslant |H|} (|h| + J_x^{(n)}(H+h)) < -\delta] \\ &\leqslant \exp\left(-\frac{\delta}{2\sigma_n^2}\right) \sum_{|h| \geqslant |H|} \exp\left(-\frac{|h|}{4\sigma_n^2}\right) \\ &\quad + \exp\left(-\frac{\delta}{\sigma_n^2}\right) \sum_{|h| \geqslant |H|} \exp\left(-\frac{|h|}{\sigma_n^2}\right) \end{aligned} \tag{3.14}$$

Observe that the inequality (3.13) introduces an exponential term in our bounds, even if we were to start from purely Gaussian ones. This is not an artefact of our estimation, but a true effect. To see why, note that

$$\sum_{|h| \geqslant |H|} \exp\left(-\frac{(\delta+|h|)^2}{2\sigma_n^2\,|H+h|}\right) \geqslant \sup_{|h| \geqslant |H|} \exp\left(-\frac{(\delta+|h|)^2}{2\sigma_n^2\,|H+h|}\right)$$
$$\geqslant \exp\left(-\frac{2(\delta-|H|)}{\sigma_n^2}\right) \tag{3.15}$$

if $\delta > 3\,|H|$. Thus, purely Gaussian bounds on the probabilities are not stable under the renormalization process, in contrast to the random bond case.

Now the sums occurring in (3.14) are all convergent and in fact bounded uniformly in σ_n (for σ_n small enough) by constants of the order of $\exp(-|H|/4\sigma_n^2)$. This yields the desired bounds for the first term in (3.11).

One might try to apply similar estimates to bound the second term in (3.11). This gives

$$\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta]$$
$$\leqslant \sum_{|h|<|H|} \mathbb{P}[|h| + J_x^{(n)}(H+h) < -\delta] \tag{3.16}$$

and using

$$\exp\left(-\frac{(\delta+|h|)^2}{2\sigma_n^2\,|H+h|}\right) \leqslant \exp\left(-\frac{\delta^2+|h|^2}{2\sigma_n^2(|H|+|h|)}\right)$$
$$\leqslant \exp\left(-\frac{\delta^2}{4\sigma_n^2\,|H|} - \frac{|h|^2}{4\sigma_n^2\,|H|}\right) \tag{3.17}$$

one would get

$$\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta]$$
$$\leqslant \exp\left(-\frac{\delta^2}{4\sigma_n^2\,|H|}\right) \sum_{|h|<|H|} \exp\left(-\frac{|h|^2}{4\sigma_n^2\,|H|}\right)$$
$$+\exp\left(-\frac{\delta}{\sigma_n^2}\right) \sum_{|h|<|H|} \exp\left(-\frac{|h|}{\sigma_n^2}\right) \tag{3.18}$$

Unfortunately, the first sum in (3.18) is of order $\sigma_n\,|H|^{1/2}$, which can be arbitrarily large. Note that this is not an artefact of the bound (3.17), but arises also if we insert the left-hand side of (3.17) into (3.16). It is thus the fact that we estimate the probability of the infimum by a sum of probabilities in (3.16) that introduces an unacceptably large error if $|H|$ is too large. For $|H|$ such that $|H|\,\sigma_n^2 \gg 1$, we must therefore improve on the estimate (3.16).

Now (3.16) would be exact if the events occurring in the infimum were disjoint; thus, to improve on it we need to use information on the correlation of these events, i.e., on the correlations between the $J_x^{(n)}(H)$ for neighboring heights H. This is another reason why we need to consider the difference process $D_x^{(n)}(H, H')$ all the time; had we started to renormalize just the original $J_x(H)$ and kept track only on bounds of the form (1.10), (1.11), as was done in the random bond model, we would be stuck at this point.

The idea to improve on (3.16) is now to group the events in the infimum into blocks in such a way that events within a block are strongly correlated, while the number of blocks is sufficiently small to allow us to bound the probability of infimum over all heights by a sum over blocks of the probabilities of the infima taken over the h in one block. To be precise, let us assume that $\sigma_n^2\,|H| > 1$. Consider a finite family of subsets $B_i \subset \mathbb{Z}$ such that their union contains $\{-H, -H+1,..., H\}$. Let $b_i \in B_i$ denote some fixed representative element of each block that we choose such that for all $h \in B_i$, $|h| \geqslant |b_i|$. Then

$$\begin{aligned}
&\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta] \\
&\quad \leqslant \sum_i \mathbb{P}[\inf_{h\in B_i} (|h| + J_x^{(n)}(H+h)) < -\delta] \\
&\quad \leqslant \sum_i \mathbb{P}[J_x^{(n)}(H+b_i) < -(\delta + |b_i|) + \varepsilon_i] \\
&\quad\quad + \sum_i \mathbb{P}[\inf_{h\in B_i} (J_x^{(n)}(H+h) - J_x^{(n)}(H+b_i) + |h| - |b_i|) < -\varepsilon_i] \qquad (3.19)
\end{aligned}$$

for arbitrary $\varepsilon_i > 0$ to be specified later. In the last line of (3.19) we can now use (3.3) to write

$$J_x^{(n)}(H+h) - J_x^{(n)}(H+b_i) = D_x^{(n)}(H+h, H+b_i) \qquad (3.20)$$

Also, by our choice of b_i, the term $|h| - |b_i|$ in (3.19) is always positive and may be dropped. Finally, estimating the probability of the infima in the last line of (3.19) by a sum, we arrive at the bound

$$\begin{aligned}
&\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta] \\
&\quad \leqslant \sum_i \Big\{ \mathbb{P}[J_x^{(n)}(H+b_i) < -(\delta + |b_i|) + \varepsilon_i] \\
&\quad\quad + \sum_{h\in B_i} \mathbb{P}[D_x^{(n)}(H+h, H+b_i) < -\varepsilon_i] \Big\} \qquad (3.21)
\end{aligned}$$

We now choose the ε_i as $\varepsilon_i = (\delta + |b_i|)/2$. Using our assumptions on the distributions (3.5), we may further write

$$\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta]$$

$$\leqslant \sum_i \left\{ \exp\left(-\frac{(\delta+|b_i|)^2}{8\sigma_n^2(|H|+|b_i|)}\right) + \exp\left(-\frac{\delta+|b_i|}{2\sigma_n^2}\right) + |B_i| \left[\exp\left(-\frac{(\delta+|b_i|)^2}{8\sigma_n^2 |B_i|}\right) + \exp\left(-\frac{\delta+|b_i|}{2\sigma_n^2}\right)\right]\right\} \quad (3.22)$$

where in the estimation of the second term in (3.21) we have used that $\max_{h \in B_i} |h - b_i| \leqslant |B_i|$. We now have to choose our family of blocks in such a way that (3.22) gives a useful bound. It turns out that for $\sigma_n |H|^{1/2} < \delta < |H|$ we may choose

$$B_i = [i[\delta], (i+1)[\delta]] \cap \mathbb{Z} \quad (3.23)$$

where $[\delta]$ denotes the largest integer less than δ. Noticing that $|b_i| \leqslant |H|$ for all blocks, $|B_i| \leqslant \delta \leqslant |H|$, and $\delta/2 \leqslant [\delta] \leqslant \delta$ (remember that $\delta \geqslant 1$), we finally obtain from (3.22)

$$\mathbb{P}[\inf_{|h|<|H|} (|h| + J_x^{(n)}(H+h)) < -\delta]$$

$$\leqslant 2 \sum_{i \geqslant 0} \left\{ \exp\left(-\frac{\delta^2(1+i/2)^2}{16\sigma_n^2 |H|}\right) + \exp\left(-\frac{\delta(1+i/2)}{2\sigma_n^2}\right) + |\delta| \left[\exp\left(-\frac{\delta^2(1+i/2)^2}{8\sigma_n^2 \delta}\right) + \exp\left(-\frac{\delta(1+i/2)}{2\sigma_n^2}\right)\right]\right\} \quad (3.24)$$

which is easily checked to be of the desired form. If δ is larger than $|H|$, we need only the one block $\{-H,..., H\}$ to get the desired result. Putting together (3.8), (3.14), (3.24), we thus obtain bounds of the form

$$\mathbb{P}[I_x(H) < -\delta] \leqslant \exp\left(-\frac{\delta^2}{2\sigma_n'^2 |H|}\right) + \exp\left(-\frac{\delta}{\sigma_n'^2}\right) \quad (3.25)$$

for all $\delta \geqslant \sigma' |H|^{1/2}$, where σ' is of the order of σ. The proof of Proposition 3.1 will now follow from the following result.

Lemma 3.2. Let $\{X_i\}_{i=1,...,N}$ be an independent family of random variables, such that, for some positive τ, where $\tau \leqslant \tau_0 < 1$:

(i) $\mathbb{E}X_i = 0$.

(ii) The following conditions hold:

$$\mathbb{P}[X_i > \delta] \leqslant \exp\left(-\frac{\delta^2}{2}\right) + \exp\left(-\frac{\delta}{\tau}\right)$$
$$\mathbb{P}[X_i < -\delta] \leqslant \exp\left(-\frac{\delta^2}{2}\right) + \exp\left(-\frac{\delta}{\tau}\right) \tag{3.26}$$

for all $\delta \geqslant 1$.

Then, if $X \equiv (1/M) \sum_{i=1}^{N} X_i$, there exists a constant $c > 1$ independent of N, M, τ, such that for all $\delta > 0$,

$$\mathbb{P}[X > \delta] \leqslant \exp(-\delta^2/2\tilde{\sigma}^2) + \exp(-\delta/\tilde{\tau})$$
$$\mathbb{P}[X < -\delta] \leqslant \exp(-\delta^2/2\tilde{\sigma}^2) + \exp(-\delta/\tilde{\tau}) \tag{3.27}$$

where

$$\tilde{\sigma} = \frac{c\sqrt{N}}{M} \qquad \text{and} \qquad \tilde{\tau} = \frac{2c^2}{M}\tau$$

Proof. To prove the lemma, we first derive from (3.26) bounds on the Laplace transform of the X_i. From these, (3.27) will follow from a standard application of the exponential Markov inequality.[19]

We will only prove the bound on $\mathbb{P}[X > \delta]$; the other one follows symmetrically. Thus, it is enough to bound the Laplace transform $\mathbb{E}e^{tX_i}$ for $t > 0$. We will distinguish the cases $t \geqslant 1$ and $t < 1$. Let first $t \geqslant 1$. We have

$$\begin{aligned}
\mathbb{E}e^{tX_i} &= t \int_{-\infty}^{\infty} e^{tx} \mathbb{P}[X_i \geqslant x]\, dx \\
&\leqslant t \int_{-\infty}^{1} e^{tx}\, dx + t \int_{1}^{\infty} e^{tx} \mathbb{P}[X \geqslant x]\, dx \\
&\leqslant e^t + t \int_{1}^{\infty} e^{tx} e^{-x^2/2}\, dx + t \int_{1}^{\infty} e^{tx} e^{-x/\tau}\, dx \\
&\leqslant e^t + (2\pi)^{1/2}\, t e^{t^2/2} + \frac{t e^{-(1/\tau - t)}}{1/\tau - t}
\end{aligned} \tag{3.28}$$

For $t < 1/c_1^2 \tau$ with any $c_1 > 1$ we have for the last term in (3.28)

$$\frac{t e^{-(1/\tau - t)}}{1/\tau - t} \leqslant \frac{1}{c_1^2 - 1}\, e^{-(1 - 1/c_1^2)(1/\tau)} \tag{3.29}$$

which is bounded uniformly by a constant, since $\tau \leqslant 1$. Since $t \geqslant 1$, we conclude that there is a constant $c_2 > 1$ such that

$$\mathbb{E}e^{tX_i} \leqslant e^{c_2^2 t^2/2} \qquad \text{for all} \quad 1 < t \leqslant 1/c_1^2 \tau \tag{3.30}$$

For $t < 1$ we use

$$\begin{aligned}\mathbb{E}e^{tX_i} &\leqslant 1 + \frac{t^2}{2}\left(\mathbb{E}[X_i^2 1_{X_i<0}] + \mathbb{E}[X_i^2 e^{tX_i} 1_{X_i \geqslant 0}]\right)\\ &\leqslant \exp\left[\frac{t^2}{2}\left(\mathbb{E}[X_i^2 1_{X_i<0}] + \mathbb{E}[X_i^2 e^{X_i} 1_{X_i \geqslant 0}]\right)\right]\end{aligned} \tag{3.31}$$

where in the last line we have estimated the second term in the argument of the exponential by its value for $t = 1$. From the bounds (3.26) we can now obtain uniform estimates on the expectations in the exponential. For example, for the second term we may write

$$\begin{aligned}&\mathbb{E}[X_i^2 e^{X_i} 1_{X_i \geqslant 0}]\\ &\quad = \int_0^\infty \frac{d}{dx}(x^2 e^x)\, \mathbb{P}[X_i \geqslant x]\, dx\\ &\quad \leqslant \int_0^1 \frac{d}{dx}(x^2 e^x)\, dx + \int_1^\infty \frac{d}{dx}(x^2 e^x)(e^{-x^2/2} + e^{-x/\tau})\, dx\end{aligned} \tag{3.32}$$

which is again bounded uniformly by a some constant, since $\tau \leqslant \tau_0$. For $\mathbb{E}[X_i^2 1_{X_i<0}]$ we proceed in the same way to see that there is a constant c_3 such that

$$\mathbb{E}e^{tX_i} \leqslant e^{c_3^2 t^2/2} \qquad \text{for all} \quad t < 1 \tag{3.33}$$

Putting together the two ranges for t and choosing $c = \max\{c_1, c_2, c_3\}$, we obtain from (3.30) and (3.33)

$$\mathbb{E}e^{tX_i} \leqslant e^{c^2 t^2/2} \qquad \text{for all} \quad t \leqslant 1/c^2 \tau \tag{3.34}$$

To prove the bounds (3.27), it is sufficient to consider only the case $M = 1$, since the general case follows by simple rescaling. For

$$Z \equiv \sum_{i=1}^{N} X_i \tag{3.35}$$

we have from the exponential Markov inequality

$$\mathbb{P}[Z > \delta] \leqslant e^{-t\delta}\mathbb{E}e^{tZ} \leqslant e^{-t\delta} e^{c^2 t^2 N/2} \qquad \text{for all} \quad t \leqslant 1/c^2 \tau \tag{3.36}$$

where we have used the independence of the X_i. The bound on the probability can now be obtained by choosing t optimal within the allowed range. For $\delta < N/\tau$ we take $t^* = \delta/c^2N < 1/c^2\tau$ to obtain

$$\mathbb{P}[Z > \delta] \leqslant e^{\delta^2/2c^2N} \tag{3.37}$$

while for $\delta \geqslant N/\tau$ with $t^* = 1/c^2\tau$ we have from (3.36)

$$\mathbb{P}[Z > \delta] \leqslant e^{\delta/2c^2\tau} \tag{3.38}$$

which concludes the proof of Lemma 3.2. ∎

The proof of Proposition 3.1 is now finished by using (3.25) and Lemma 3.2 for the independent family of centered random variables

$$X_x \equiv \frac{1}{\sigma'_n |H|^{1/2}} \left[\inf_{h \in \mathbb{Z}} (|h| + J_x^{(n)}(H+h)) - \inf_{h \in \mathbb{Z}} (|h| + J_x^{(n)}(h))\right] \tag{3.39}$$

for $x \in \mathscr{L}y$, and with $M = L^{d-1}$. Note that the variance of the sum variables is then scaled by a factor $cL^{d/2-d+1}$, which can be made smaller than one by choosing L sufficiently large, *provided* $d > 2$. It is at this point only where the dimensionality of the system enters the proof. ∎

We are now ready to estimate the expectation of the height $m_x(J) \equiv m_x(\infty, J)$ of the surface and thus to prove Theorem 1 in the special case $\beta = \infty$, that is, we show the following result.

Proposition 3.3. Let $\{J_x(H)\}_{H \in \mathbb{Z}}$ be stochastic processes as specified in Theorem 1. Then there exist $\sigma_0 > 0$, $L_0 < \infty$, and a finite, positive constant c such that for all $\sigma \leqslant \sigma_0$ and $L \geqslant L_0$,

$$\mathbb{P}[m_x(J) > \delta] \leqslant \exp(-\delta/\tilde{\sigma}^2) \tag{3.40}$$

for all $\delta > 0$ and for all $x \in \mathbb{Z}^d$, where $\tilde{\sigma} = c\sigma$.

Proof. Our proof will use, of course, the recursive scheme introduced in Section 2. We will first show that for all N, H and for all $\delta > 0$

$$\mathbb{P}[\langle |h_y^{(n)}| \rangle_N (H) > \delta] \leqslant \exp(-\delta/\sigma_n'^2) \tag{3.41}$$

with $\sigma'_n = c'\sigma_n$, where c' is a constant independent of σ, n, and N. These bounds can then easily be summed over n to yield (3.40).

In order to prove (3.41), we have to control the recursion relation (2.18). For zero temperature, this recursion simplifies considerably, since then the sum in (2.18) is then given by only one term. Let thus $h_y^{*(n)}(H)$

denote the random variable with values in $\mathbb{Z}$ which is given by the h realizing the inf in the definition of $I_y(H)$:

$$\inf_{h \in \mathbb{Z}} (|h| + J_y^{(n)}(H+h)) = |h_y^{*(n)}(H)| + J_y^{(n)}[H + h_y^{*(n)}(H)] \tag{3.42}$$

It follows easily from the bounds we obtain from Proposition 3.1 that a solution of (3.42) exists almost surely. Under some weak assumption on the distributions of the $J_x(H)$ (continuity is, for instance, a sufficient condition) it is unique a.s., which we will assume here. (This assumption is, however, not really necessary and we will not make it in the proof of Theorem 1 given in the next section.)

We may now rewrite (2.18) in the case $\beta = \infty$ as

$$\begin{aligned} &\langle |h_y^{(n)}| \rangle_n (H) \equiv |h_y^{*(m)}(H)| \\ &\langle |h_y^{(n)}| \rangle_m (H) \equiv |h_y^{*(m)}(H)| + \langle |h_y^{(n)}| \rangle_{(m-1)} [H + h_y^{*(m)}(H)] \quad \text{for } m > n \end{aligned} \tag{3.43}$$

Note that, due to stationarity, the $\langle |h_y^{(n)}| \rangle_m (H)$ have the same distribution for all $H \in \mathbb{Z}$. Thus we begin the estimation of (3.43) with

$$\begin{aligned} \mathbb{P}[|h_y^{*(n)}(H)| > \delta] &\leqslant \mathbb{P}[\exists h \in \mathbb{Z}, |h| > \delta \colon |h| + J_y^{(n)}(H+h) < J_y^{(n)}(H)] \\ &\leqslant \sum_{|h| > \delta} \mathbb{P}[D_y^{(n)}(H+h, H) < -|h|] \\ &\leqslant \sum_{|h| > \delta} \left[\exp\left(- \frac{h^2}{2\sigma_n^2 |h|} \right) + \exp\left(- \frac{|h|}{\sigma_n^2} \right) \right] \\ &\leqslant \exp\left(- \frac{\delta}{2\bar{\sigma}_n^2} \right) \end{aligned} \tag{3.44}$$

where $\bar{\sigma}_n = c_1 \sigma_n$ (with $c_1 > 1$ being some constant). This bound is already of the form (3.41). Next, we introduce a partition of our probability space according to the value that $h_y^{*(n)}(H)$ takes, that is, we write the second line in (3.43) as

$$\begin{aligned} &\mathbb{P}[\langle |h_y^{(n)}| \rangle_m (H) > \delta] \\ &\quad \leqslant \mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H) > \delta \wedge h_y^{*(m)}(H) = 0] \\ &\quad + \sum_{h \neq 0} \mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H+h) > \delta \wedge h_y^{*(m)}(H) = h] \end{aligned} \tag{3.45}$$

Now we use the simple fact that $\mathbb{P}[A \cap B] \leqslant \min\{\mathbb{P}[A], \mathbb{P}[B]\}$ and the estimate [see (3.44)]

$$\mathbb{P}[h_y^{*(m)}(H) = h] \leqslant \exp(-h/\bar{\sigma}_m^2) \tag{3.46}$$

to arrive at

$$\begin{aligned}
&\mathbb{P}[\langle |h_y^{(n)}| \rangle_m (H) > \delta] \\
&\quad \leqslant \mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H) > \delta] \\
&\qquad + \sum_{h \neq 0} \min\{\mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H+h) > \delta], \exp(-h/\bar{\sigma}_m^2)\} \qquad (3.47)
\end{aligned}$$

All that is now left to do is to iterate this bound, starting with $m = n$ up to $m = N$, the initial bound given by (3.42). This is the subject of the following lemma:

Lemma 3.4. Let the sequence $\{q_k\}_{k \geqslant 0}$ be recursively defined by

$$q_{k+1} = q_k + \sum_{h \neq 0} \min\left\{q_k, \exp\left(-\frac{|h|}{\gamma_k^2}\right)\right\} \qquad (3.48)$$

where for some $0 < c < 1$

$$\gamma_k = c^k \gamma \qquad (3.49)$$

Then, if γ and q_0 are sufficiently small, there is a constant $0 < c' < 1$, independent of γ and q_0, such that

$$q_k \leqslant q_0^{c'} \qquad (3.50)$$

Assuming this lemma, the proof of Proposition 3.3 is now finished: Just note that if we set

$$q_0 = \mathbb{P}[|h_y^{*(n)}(H)| > \delta], \qquad \gamma_k = \bar{\sigma}_{n+k+1} = c^k c_1 c^{n+1} \sigma \qquad (3.51)$$

then (3.47) implies that

$$\mathbb{P}[\langle |h_y^{(n)}| \rangle_{n+k} (H) > \delta] \leqslant q_k \qquad (3.52)$$

and thus

$$\mathbb{P}[\langle |h_y^{(n)}| \rangle_m (H) > \delta] \leqslant \exp(-c'\delta/2\bar{\sigma}_n^2) \qquad (3.53)$$

since $\bar{\sigma}_n$ and subsequently q_0 are assumed to be small. This concludes the proof of Proposition 3.3. ∎

Let us finally give the proof of Lemma 3.4:

Proof. Noticing that q_k is increasing while γ_k is going to zero under iteration of (3.48), it is clear that there will be a k_0 such that for $k \geqslant k_0$ for all h in the sum the minimum will be given by the second term. We will have to estimate (3.48) for $k \geqslant k_0$, for $k < k_0$, and estimate the value of k_0 itself.

If we assume that $j \geqslant k_0$, i.e., $q_k \geqslant \exp(-1/\gamma_k^2)$, we have from the iteration of (3.48) for all $m \geqslant j$

$$\begin{aligned} q_m &\leqslant q_j + 2 \sum_{l=j}^{m-1} \sum_{h=1}^{\infty} \exp\left(-\frac{h}{\gamma_l^2}\right) \\ &= q_j + 2 \sum_{l=j}^{m-1} \frac{1}{\exp(1/\gamma_l^2) - 1} \\ &\leqslant q_j + O\left(\exp\left(-\frac{1}{\gamma_j^2}\right)\right) \\ &\leqslant c_2 q_j \end{aligned} \tag{3.54}$$

for γ sufficiently small with some constant $c_2 > 1$. Hence, if q_0 is larger than $\exp(-1/\gamma^2)$, we have already proven (3.41).

Let us now assume $q_0 \leqslant \exp(-1/\gamma^2)$. Obviously, the number of terms in the sum (3.48) for which q_k realizes the minimum can be computed simply by solving $q_0 = \exp(-h/\gamma_j^2)$ for h, so that

$$\begin{aligned} q_{j+1} &\leqslant q_j + 2\#\left\{h \geqslant 1, q_j \leqslant \exp\left(-\frac{1}{\gamma_j^2}\right)\right\} + c_2 q_j \\ &\leqslant c_3(1 - \gamma_j^2 \ln q_j)\, q_j \end{aligned} \tag{3.55}$$

with some new constant $c_3 > 1$.

Now, since q_k is increasing, k_0 is bounded by the minimal k such that $q_0 \leqslant \exp(-1/\gamma_k^2)$, which we denote by $\bar{k}$. Estimating the q_j in the logarithm of (3.55) by q_0, it follows that

$$q_{\bar{k}} \leqslant c_3^{\bar{k}} \prod_{l=1}^{\bar{k}} (1 - \gamma_l^2 \ln q_0)\, q_0 \tag{3.56}$$

To bound the terms appearing in this expression, note first that by the definition of $\bar{k}$ we see that for γ small enough

$$c_3^{\bar{k}} \leqslant \left(\ln \frac{1}{q_0}\right)^{c_4} \tag{3.57}$$

with $c_4 > 0$. Further, with yet another constant $c_5 > 0$,

$$\begin{aligned} \prod_{l=1}^{\bar{k}} (1 - \gamma_l^2 \ln q_0) &\leqslant \exp\left[\sum_{l=1}^{\infty} \ln(1 - \gamma_l^2 \ln q_0)\right] \\ &\leqslant \exp\left(-\sum_{l=1}^{\infty} \gamma_l^2 \ln q_0\right) \leqslant q_0^{-c_5 \gamma^2} \end{aligned} \tag{3.58}$$

Finally, putting together (3.54) and (3.56)–(3.58), we get for all $k \geqslant 0$

$$q_k \leqslant c_2 \left(\ln \frac{1}{q_0} \right)^{c_4} q_0^{1 - c_5 \gamma^2}$$
$$\leqslant q_0^{c_6^2} \tag{3.59}$$

with $0 < c_6 < 1$, which is valid for γ and q_0 sufficiently small. This proves Lemma 3.4. ∎

4. THE RENORMALIZATION GROUP FLOW AT FINITE TEMPERATURE

In this section we present the complete proof of Theorem 1, that is, we treat the case $\beta < \infty$. The structure of the proof is essentially the same as in the previous section and we will make frequent reference to it, while stressing the points that require modifications.

We begin by stating the analogue of Proposition 3.1, as follows.

Proposition 4.1. Let $\{D_x^{(n)}(H, H')\}_{H, H' \in \mathbb{Z}}$, for $x \in \mathbb{Z}^d$, be as specified in Proposition 3.1, i.e., centered, stationary difference processes obeying

$$\mathbb{P}[D_x^{(n)}(H, H') < -\delta] \leqslant \exp\left(- \frac{\delta^2}{2\sigma_n^2 \, |H - H'|} \right) + \exp\left(- \frac{\delta}{\sigma_n^2} \right) \tag{4.1}$$

Then, $\{D_x^{(n+1)}(H, H')\}_{H, H' \in \mathbb{Z}}$ are again centered stationary difference processes and, for $d > 2$, if L and β_n are large enough and σ_n small enough, there exists a constant $c < 1$, independent of L, $\beta^{(n)}$, and σ_n such that with $\sigma_{n+1} = c\sigma_n$, for all $\delta > 0$,

$$\mathbb{P}[D_x^{(n+1)}(H, H') < -\delta] \leqslant \exp\left(- \frac{\delta^2}{2\sigma_{n+1}^2 \, |H - H'|} \right) + \exp\left(- \frac{\delta}{\sigma_{n+1}^2} \right) \tag{4.2}$$

Proof. That $D_x^{(n+1)}(H, H')$ are stationary, centered difference processes follows from the same arguments as in the proof of Proposition 3.1.

To show the bounds (4.2) for the renormalized process, it suffices thus again to consider $D_y^{(n+1)}(H, 0) \equiv J_y^{(n+1)}(H)$. Exactly as in Section 2, we obtain for them the inequality

$$J_y^{(n+1)}(H) \geqslant \frac{1}{L^{d-1}} \sum_{x \in \mathscr{L}y} \Phi_{\beta^{(n)}}(\{|h| + J_x^{(n)}(H + h)\}_{h \in \mathbb{Z}}) \tag{4.3}$$

We thus are left to control the distribution of the variables $\Phi_{\beta^{(n)}}(\{|h| + J_x^{(n)}(H+h)\}_{h\in\mathbb{Z}})$, for which we separate again the contributions of "small" and "large" h's to the sum:

$$\begin{aligned}
&\mathbb{P}[\Phi_{\beta^{(n)}}(\{|h| + J_x^{(n)}(H+h)\}_{h\in\mathbb{Z}}) < -\delta] \\
&\quad = \mathbb{P}\left[\sum_{h\in\mathbb{Z}} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > e^{\beta^{(n)}\delta}\right] \\
&\quad \leqslant \mathbb{P}\left[\sum_{|h|\geqslant|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > pe^{\beta^{(n)}\delta}\right] \\
&\qquad + \mathbb{P}\left[\sum_{|h|<|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\, e^{\beta^{(n)}\delta}\right] \qquad (4.4)
\end{aligned}$$

for any $0<p<1$. To bound the probability of the sum by the sum of probabilities, we introduce the constants (this choice of constants is to a large degree arbitrary and not necessarily optimal)

$$\alpha_h \equiv \frac{e^{-|h|}}{K} \qquad (4.5)$$

with K chosen such that $\sum_{h\in\mathbb{Z}} \alpha_h = 1$. For the first term in (4.4) we write

$$\begin{aligned}
&\mathbb{P}\left[\sum_{|h|\geqslant|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > pe^{\beta^{(n)}\delta}\right] \\
&\quad \leqslant \sum_{|h|\geqslant|H|} \mathbb{P}[\exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > p\alpha_h e^{\beta^{(n)}\delta}] \\
&\quad \leqslant \sum_{|h|\geqslant|H|} \mathbb{P}\left[J_x^{(n)}(H+h) < -\delta - \left(1-\frac{1}{\beta^{(n)}}\right)|h| + \frac{1}{\beta^{(n)}}\ln\left(\frac{K}{p}\right)\right] \qquad (4.6)
\end{aligned}$$

We note that, given any fixed $0<c_1<1$,

$$\left(1-\frac{1}{\beta^{(n)}}\right)|h| - \frac{1}{\beta^{(n)}}\ln\left(\frac{K}{p}\right) \geqslant c_1\,|h| \qquad (4.7)$$

for all $h\neq 0$, if

$$\beta^{(n)} \geqslant \frac{\ln(K/p)-1}{1-c_1} \qquad (4.8)$$

That means that for such $\beta^{(n)}$ we have

$$\begin{aligned}
&\mathbb{P}\left[\sum_{|h|\geqslant|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > pe^{\beta^{(n)}\delta}\right] \\
&\quad \leqslant \sum_{|h|\geqslant|H|} \mathbb{P}[J_x^{(n)}(H+h) < -(\delta + c_1\,|h|)] \qquad (4.9)
\end{aligned}$$

Clearly, in (4.6) the temperature dependence has manifested itself only in that the constant c_1 has become slightly smaller than 1. Obviously this leaves us now with the same computations as in the case $\beta = \infty$.

To estimate the second term in (4.4), we distinguish as in Section 2 the cases of large and small $|H|$: In the case $\sigma_n^2 |H| \leqslant 1$ we write, as in (4.9),

$$\mathbb{P}\left[\sum_{|h|<|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\, e^{\beta^{(n)}\delta}\right] \leqslant \sum_{|h|<|H|} \mathbb{P}[J_x^{(n)}(H+h) < -(\delta + c_1 |h|)] \tag{4.10}$$

From here on we may proceed as in (3.17) and (3.18).

For $\sigma_n^2 |H| > 1$ we use the blocking with the same family of blocks $B_i \subset \mathbb{Z}$ as in Section 3. Then

$$\begin{aligned}
&\mathbb{P}\left[\sum_{|h|<|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\, e^{\beta^{(n)}\delta}\right] \\
&\leqslant \sum_i \mathbb{P}\left[\sum_{h \in B_i} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\, \alpha_i e^{\beta^{(n)}\delta}\right] \\
&\leqslant \sum_i \mathbb{P}\left[|b_i| + J_x^{(n)}(H+b_i) \right.\\
&\qquad - \frac{1}{\beta^{(n)}} \ln \sum_{h \in B_i} \exp\{-\beta^{(n)}(J_x^{(n)}(H+h) - J_x^{(n)}(H+b_i) + |h| - |b_i|)\} \\
&\qquad \left. < -\delta - \frac{\ln((1-p)\,\alpha_i)}{\beta^{(n)}}\right]
\end{aligned} \tag{4.11}$$

with α_i being some new positive constants with $\sum \alpha_i = 1$. Now we separate the random variables on the l.h.s in the probability by introducing $\varepsilon_i \geqslant 0$,

$$\begin{aligned}
&\mathbb{P}\left[\sum_{|h|<|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\, e^{\beta^{(n)}\delta}\right] \\
&\leqslant \sum_i \left\{\mathbb{P}\left[J_x^{(n)}(H+b_i) < -\delta - |b_i| - \frac{\ln((1-p)\,\alpha_i)}{\beta^{(n)}} + \varepsilon_i\right]\right. \\
&\qquad \left. + \mathbb{P}\left[\sum_{h \in B_i} \exp\{-\beta^{(n)} D_x^{(n)}(H+h, H+b_i)\} > e^{\beta^{(n)}\varepsilon_i}\right]\right\} \\
&\leqslant \sum_i \left\{\mathbb{P}\left[J_x^{(n)}(H+b_i) < -\delta - |b_i| - \frac{\ln((1-p)\,\alpha_i)}{\beta^{(n)}} + \varepsilon_i\right]\right. \\
&\qquad \left. + \sum_{h \in B_i} \mathbb{P}[D_x^{(n)}(H+h, H+b_i) < -\varepsilon_i] + \frac{\ln |B_i|}{\beta^{(n)}}\right\}
\end{aligned} \tag{4.12}$$

We now choose the ε_i such that the r.h.s. of the inequalities in the probabilities in (4.11) are the same, that is,

$$\varepsilon_i = \frac{1}{2}\left(\delta + |b_i| + \frac{\ln((1-p)\,\alpha_i\,|B_i|)}{\beta^{(n)}}\right) \tag{4.13}$$

and further set

$$\alpha_i = e^{-|b_i|}/K \tag{4.14}$$

to arrive at

$$\begin{aligned}
&\mathbb{P}\left[\sum_{|h|<|H|} \exp\{-\beta^{(n)}(|h| + J_x^{(n)}(H+h))\} > (1-p)\,e^{\beta^{(n)}\delta}\right] \\
&\quad \leqslant \sum_i \left\{\mathbb{P}\left[J_x^{(n)}(H+b_i) < -\frac{1}{2}\left\{\delta + \left(1 - \frac{1}{\beta^{(n)}}\right)|b_i| - \frac{1}{\beta^{(n)}}\ln\frac{K\,|B_i|}{1-p}\right\}\right]\right. \\
&\qquad + \sum_{h\in B_i} \mathbb{P}\left[D_x^{(n)}(H+h,\, H+b_i)\right. \\
&\qquad \left.\left. < -\frac{1}{2}\left\{\delta + \left(1 - \frac{1}{\beta^{(n)}}\right)|b_i| - \frac{1}{\beta^{(n)}}\ln\frac{K\,|B_i|}{1-p}\right\}\right]\right\} \\
&\quad \leqslant \sum_i \left\{\mathbb{P}\left[J_x^{(n)}(H+b_i) < -\frac{1}{2}(\delta + c_1\,|b_i| - c_2\,|B_i|)\right]\right. \\
&\qquad \left. + \sum_{h\in B_i} \mathbb{P}\left[D_x^{(n)}(H+h,\, H+b_i) < -\frac{1}{2}(\delta + c_1\,|b_i| - c_2\,|B_i|)\right]\right\}
\end{aligned} \tag{4.15}$$

with $0 < c_2 < 1$ for $\beta^{(n)}$ large enough. This leaves us again in a similar situation as in the zero-temperature case.

The proof of Proposition 4.1 now follows by putting together (4.9), (4.10), and (4.15) and finally applying Lemma 3.2. ▮

With Proposition 4.1 we now have control of the renormalization group flow on the random variables $D^{(n)}$. To complete the proof of Theorem 1, we now just have to prove the analogue of Proposition 4.3 for the case of finite temperature.

To do this, we again have to control the recursion (2.18). Again this will be done by first estimating the initial value, then deriving a recursive inequality analogous to (3.47), and finally proving a lemma corresponding to Lemma 3.4 which will allow us to control it.

We begin with the first point

$$\begin{aligned}
&\mathbb{P}[\langle |h_y^{(n)}|\rangle_n (H) > \delta] \\
&\quad = \mathbb{P}\left[\frac{\sum_h |h| \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\}}{\sum_h \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\}} > \delta\right] \\
&\quad = \mathbb{P}\Bigg[\sum_{|h|>\delta} (|h|-\delta) \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\} \\
&\quad > \sum_{|h|\leqslant\delta} (\delta - |h|) \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\}\Bigg] \\
&\quad \leqslant \mathbb{P}\Bigg[\sum_{|h|>\delta} (|h|-\delta) \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\} \\
&\quad > \delta \exp[-\beta^{(n)} J_y^{(n)}(H)]\Bigg] \\
&\quad \leqslant \sum_{|h|>\delta} \mathbb{P}[(|h|-\delta) \exp\{-\beta^{(n)}[|h| + J_y^{(n)}(H+h)]\} \\
&\quad > \alpha_h \delta \exp[-\beta^{(n)} J_y^{(n)}(H)]] \\
&\quad \leqslant \sum_{|h|>\delta} \mathbb{P}\left[D_y^{(n)}(H+h, H) < -|h| + \frac{1}{\beta^{(n)}} \ln \frac{|h|-\delta}{\alpha_h \delta}\right] \qquad (4.16)
\end{aligned}$$

Choosing the constants $\alpha_h \geqslant 0$ with $\sum \alpha_h = 1$ as in (4.5), we find for the l.h.s. of the inequality in the probability in the last line of (4.16) that for any $0 < c_1 < 1$

$$-|h| + \frac{1}{\beta^{(n)}} \ln \frac{|h|-\delta}{\alpha_h \delta} \leqslant -c_1 |h| \qquad (4.17)$$

if

$$\delta \geqslant K \max_{h=1,2,\dots} \frac{h}{1 + Ke^{(\beta^{(n)}(1-c_1)-\alpha)h}} \qquad (4.18)$$

From (4.16)–(4.18) we have that for $\beta^{(n)}$ sufficiently large there is a constant $c_2 > 0$ such that for $\delta \geqslant e^{-c_2 \beta^{(n)}}$,

$$\mathbb{P}[\langle |h_y^{(n)}|\rangle_n (H) > \delta] \leqslant \sum_{|h|>\delta} \mathbb{P}[D_y^{(n)}(H+h, H) < -c_1 |h|] \qquad (4.19)$$

From (4.19) we obtain as usual that there is a constant $c_3 > 1$ such that

$$\mathbb{P}[\langle |h_y^{(n)}|\rangle_n (H) > \delta] \leqslant \min\left\{\exp\left(-\frac{1}{2\bar{\sigma}_n^2}\right), \exp\left(-\frac{\delta}{2\bar{\sigma}_n^2}\right)\right\} \qquad (4.20)$$

for $\delta \geqslant e^{-c_2 \beta^{(n)}}$, where $\bar{\sigma}_n = c_3 \sigma_n$.

To tackle the second task and derive the renormalization group inequality, we first use a similar manipulation as in (3.47):

$$\mathbb{P}[\langle |h_y^{(n)}| \rangle_m (H) > \delta]$$
$$= \mathbb{P}\left[\sum_{h \in \mathbb{Z}} (\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta) \times \exp\{-\beta^{(m)}[|h| + D_y^{(m)}(H+h, H)]\} > 0\right] \tag{4.21}$$

Next we separate the term for $h = 0$ from the rest of the sum and write, for positive η to be chosen later,

$$\begin{aligned}
\mathbb{P}[\langle |h_y^{(n)}| \rangle_m (H) > \delta]
&\leqslant \mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta > -\eta] \\
&\quad + \mathbb{P}\left[\sum_{h \neq 0} (\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta) \times \exp\{-\beta^{(m)}[|h| + D_y^{(m)}(H+h, H)]\} > \eta\right] \\
&\leqslant \mathbb{P}[\langle |h_y^{(n)}| \rangle_{m-1} (H) > \delta - \eta] \\
&\quad + \sum_{h \neq 0} \mathbb{P}[(\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta) \times \exp\{-\beta^{(m)}[|h| + D_y^{(m)}(H+h, H)]\} > \alpha_h \eta]
\end{aligned} \tag{4.22}$$

where $\alpha_h > 0$, $\sum_{h \neq 0} \alpha_h = 1$. Now, for arbitrary $\theta_h > 0$,

$$\begin{aligned}
&\mathbb{P}[(\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta) \exp\{-\beta^{(m)}[|h| + D_y^{(m)}(H+h, H)]\} > \alpha_h \eta] \\
&\quad \leqslant \mathbb{P}\Bigg[\langle |h_y^{(n)}| \rangle_{m-1} (H) > \delta \\
&\qquad \wedge \Bigg(\langle |h_y^{(n)}| \rangle_{m-1} (H) - \delta > \theta_h \\
&\qquad \vee \exp\{-\beta^{(m)}[|h| + D_y^{(m)}(H+h, H)]\} > \frac{\alpha_h \eta}{\theta_h}\Bigg)\Bigg]
\end{aligned} \tag{4.23}$$

Using $\mathbb{P}[A \cap B] \leqslant \min\{\mathbb{P}[A], \mathbb{P}[B]\}$, we finally arrive at the desired recursion inequality which corresponds to (3.47):

$$\begin{aligned}
&\mathbb{P}[\langle|h_y^{(n)}|\rangle_n(H)>\delta] \\
&\quad\leqslant \mathbb{P}[\langle|h_y^{(n)}|\rangle_{m-1}(H)>\delta-\eta] \\
&\qquad+\sum_{h\neq 0}\min\Big\{\mathbb{P}[\langle|h_y^{(n)}|\rangle_{m-1}(H+h)>\delta], \\
&\qquad\quad \mathbb{P}[\langle|h_y^{(n)}|\rangle_{m-1}(H+h)>\delta+\theta_h] \\
&\qquad+\mathbb{P}\left[D_y^{(m)}(H+h,H)<-|h|-\frac{1}{\beta^{(m)}}\ln\frac{\alpha_h\eta}{\theta_h}\right]\Big\}
\end{aligned}\tag{4.24}$$

Note that, if $\beta^{(m)}$ goes to infinity, we can send η to zero and θ_h to infinity to recover (3.47). What remains to be done is to choose the constants depending on $\beta^{(m)}$ in a suitable way and to conclude the proof along the same lines as in Section 3.

To be specific, let us set

$$\begin{aligned}
\eta_m &= e^{-c_1\beta^{(m)}} \\
\theta_h^{(m)} &= e^{c_2\beta^{(m)}|h|} \\
\alpha_h &= \frac{e^{-|h|}}{K}
\end{aligned}\tag{4.25}$$

with $0<c_1,\ c_2<1$. Thus, we may write

$$\begin{aligned}
&\mathbb{P}\left[D_y^{(m)}(H+h,H)<-|h|-\frac{1}{\beta^{(m)}}\ln\frac{\alpha_h\eta}{\theta_h}\right] \\
&\quad\leqslant \mathbb{P}\left[D_y^{(m)}(H+h,H)<-\left(1-\frac{1}{\beta^{(m)}}-c_2\right)|h|+\frac{K}{\beta^{(m)}}+c_1\right] \\
&\quad\leqslant \mathbb{P}[D_y^{(m)}(H+h,H)<-c_1'|h|] \\
&\quad\leqslant \exp\left(-\frac{|h|}{c_2'\sigma_m^2}\right)
\end{aligned}\tag{4.26}$$

with some constants $0<c_1'<1$, $c_2'\geqslant 1$, if $\beta^{(m)}$ sufficiently large and σ_m sufficiently small.

To conclude the proof of Theorem 1 and estimate the renormalization group inequality (4.24), it is thus obviously sufficient to show the following generalization of Lemma 3.4.

Lemma 4.2. Let the sequence of functions $\{q_k(\delta)\}_{k\geqslant 0}$ be recursively defined by

$$q_0(\delta)=\exp\left(-\frac{1}{\gamma^2}\max\{1,\delta\}\right)\tag{4.27}$$

for $\delta \geqslant \exp(-\bar{c}\beta_0)$ and

$$q_{k+1}(\delta) = q_k(\delta - e^{-c_1\beta_k}) + \sum_{h \neq 0} \min\{q_k(\delta), q_k(\delta + e^{c_2\beta_k |h|}) + e^{-|h|/\gamma_k^2}\} \qquad (4.28)$$

for $\delta \geqslant \exp(-\bar{c}\beta_0) + \sum_{l=0}^{k} \exp(-c_1\beta_l)$, where

$$\gamma_k = c^k\gamma, \qquad \beta_k = C^k\beta_0 \qquad (4.29)$$

and $\bar{c}$, c_1, $c_2 > 0$, $0 < c < 1$, $C > 1$ are constants.

Then, if $\gamma > 0$ is sufficiently small and $\beta_0 > 0$ sufficiently large, there are constants $0 < c' < 1$, $\bar{c}' > 0$, independent of γ and β_0, such that

$$q_k(\delta) \leqslant \exp\left(-\frac{c'}{\gamma^2}\max\{1, \delta\}\right) \qquad (4.30)$$

if $\delta \geqslant \exp(-\bar{c}'\beta_0)$.

Proof. Since $q_k(\delta)$ is increasing while γ_k goes to zero with k going to infinity, there will be $k_0(\delta)$ such that for $k \geqslant k_0(\delta)$ for all h in the sum the minimum will be given by the second term. Hence, we will again estimate (3.48) for $k \geqslant k_0(\delta)$, for $k < k_0(\delta)$, and estimate the value of $k_0(\delta)$ itself.

If we first assume $q_j(\delta) \geqslant \exp(-1/\gamma_j^2)$, it follows that

$$q_{j+1}(\delta) \leqslant q_j(\delta - e^{-c_1\beta_j}) + \sum_{h \neq 0} 2e^{-|h|/\gamma_j^2} \qquad (4.31)$$

where we have used the rough estimation

$$q_k(\delta + e^{c_2\beta_k |h|}) \leqslant q_k(\delta)$$

in the minimum, which is correct, since under iteration the $q_k(\delta)$ remain of course decreasing functions of δ. From the iteration of (4.31) we have likewise as in (3.54) for all $m \geqslant j$

$$\begin{aligned} q_m(\delta) &\leqslant q_j\left(\delta - \sum_{l=j}^{m-1} e^{-c_1\beta_j}\right) + 4\sum_{l=j}^{m-1}\sum_{h=1}^{\infty} e^{-h/\gamma_j^2} \\ &\leqslant \tilde{c}_2 q_j\left(\delta - \sum_{l=j}^{m-1} e^{-c_1\beta_j}\right) \end{aligned} \qquad (4.32)$$

for γ sufficiently small with some constant $\tilde{c}_2 > 1$, if $\delta \geqslant \exp(-\bar{c}\beta_0) + \sum_{l=0}^{m-1} \exp(-c_1\beta_l)$.

Second, we consider the case $q_j(\delta) \leqslant \exp(-1/\gamma_j^2)$ and proceed as in Section 3: We break the sum into two parts and estimate the minimum in the first part by $q_k(\delta)$, in the second part by the second term. Let us for a moment *suppose* that we already knew that

$$q_k(\delta + e^{c_2\beta_k |h|}) \leqslant e^{-|h|/\gamma_k^2} \qquad (4.33)$$

for arbitrary k. We would then be in a position to estimate

$$\begin{aligned} q_{j+1}(\delta) &\leqslant q_j(\delta - \exp(-c_1\beta_j)) \\ &\quad + 2\# \left\{h \geqslant 1, q_j(\delta) \leqslant 2\exp\left(-\frac{h}{\gamma_j^2}\right)\right\} q_j(\delta) + \tilde{c}_2 q_{j+1}(\delta) \\ &\leqslant \tilde{c}_3(1 - \gamma_j^2 \ln q_j(\delta))\, q_j(\delta - \exp(-c_1\beta_j)) \end{aligned} \tag{4.34}$$

(with some new constant $\tilde{c}_3$), which can be estimated along the lines of the zero-temperature case: Let us again denote by $\bar{k}$ the minimal number of iterations such that $q_0(\delta) \leqslant \exp(-1/\gamma_{\bar{k}}^2)$, so that we can use (4.32) with $j = \bar{k}$. Then we may obtain from (4.34)

$$q_k(\delta) \leqslant \tilde{c}_3^{\bar{k}} \prod_{l=1}^{\bar{k}} [1 - \gamma_l^2 \ln q_0(\delta)]\, q_0\left(\delta - \sum_{l=j}^{\bar{k}-1} e^{-c_1\beta_j}\right) \tag{4.35}$$

The estimation of the first two factors in (4.35) is the same as in the zero-temperature case. Now we can conclude from (4.32) and (4.35) that, with some constants $\tilde{c}_4$, $\tilde{c}_5 > 0$, for all $k \geqslant 0$,

$$\begin{aligned} q_k(\delta) &\leqslant c_2' \left(\ln \frac{1}{q_0(\delta)}\right)^{\tilde{c}_4} q_0(\delta)^{-\tilde{c}_5\gamma^2} q_0\left(\delta - \sum_{l=j}^{\infty} \exp(-c_1\beta_j)\right) \\ &\leqslant \exp\left(-\frac{c'}{\gamma^2}\max\{1, \delta\}\right) \end{aligned} \tag{4.36}$$

if $\delta \geqslant \exp(-\bar{c}'\beta_0)$ with some constants $0 < c' < 1$, $\bar{c}' > 0$, which is valid for γ and q_0 sufficiently small.

What remains is to justify the assumption (4.33). But to do so, we can perform an induction by reconsidering the old lines: (4.33) is true for $k = 0$. Suppose (4.33) holds for k; then we are allowed to use (4.34) one more time to get a bound on $q_{k+1}(\delta)$. This bound is obviously estimated by the l.h.s. of (4.36). Thus we have

$$\begin{aligned} q_{k+1}(\delta + \exp(c_2\beta_{k+1}\,|h|)) &\leqslant q_{k+1}(\exp(c_2\beta_{k+1}\,|h|)) \\ &\leqslant \exp\left(-\frac{c'}{\gamma^2} e^{c_2\beta_{k+1}|h|}\right) \\ &\leqslant \exp\left(-\frac{|h|}{\gamma_{k+1}^2}\right) \end{aligned} \tag{4.37}$$

for all $|h| \geqslant 1$, with c' being uniform in k, which proves (4.33). ∎

5. CONCLUSIONS

We have investigated a hierarchical model for a domain wall in a random-field Ising model. In this approximation exact renormalization group transformations could be performed on the stochastic processes describing the random fields. In dimension $D > 3$, we have shown that starting at weak disorder and at low temperatures, we are driven to a fixpoint of zero temperature and delta function at zero distribution for the random fields. Moreover, the speed of convergence could be controlled precisely, which allowed us to prove rigorously that in this situation a rigid interface exists.

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